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The Renormalization Group and Optimization of Entropy

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We illustrate the possible connection that exists between the extremal properties of entropy expressions and the renormalization group (RG) approach when applied to systems with scaling symmetry. We consider three examples: (1) Gaussian fixed-point criticality in a fluid or in the capillary-wave model of an interface; (2) Lévy-like random walks with self-similar cluster formation; and (3) long-ranged bond percolation. In all cases we find a decreasing entropy function that becomes minimum under an appropriate constraint at the fixed point. We use an equivalence between random-walk distributions and order-parameter pair correlations in a simple fluid or magnet to study how the dimensional anomaly at criticality relates to walks with long-tailed distributions.

KEY WORDS: Renormalization group; entropy; Gaussian model; random walks; bond percolation.

1. INTRODUCTION

Since the introduction of the renormalization group (RG) concepts in the study of critical phenomena,[1,2] and their success in explaining scaling and universality, many authors have accomplished a vast number of fruitful applications of the resultant method, first within the field of phase transitions in statistical mechanics, and then in other areas of physics, in condensed matter problems, in non-linear dynamics and in other fields.[3] The discussion of the roots and connections of the RG strategy for handling problems involving many length scales with quantum field theory has also had a long history.[3,4] Nowadays the RG theory has matured as the leading computational technique for determining the properties of systems exhibiting self-similarity under rescaling, and, after all the years of

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experience, an appraisal of its general nature has become more clearly recognized. A characteristic skill or proficiency element stands out in the RG implementations, since for every chosen construction the RG transformation can be suitably or poorly designed, inasmuch as the practical goal of obtaining critical RG flow lines that terminate at meaningful non-trivial fixed points is dependent on the particular choices made.[3] Specifically, and according to our knowledge, we note that in this formalism the guiding character of a variational approach is apparently absent.[5]

Here we recount and provide additional evidence that supports the possibility for an existing connection between the extremal properties of entropy expressions and the RG approach when applied to systems with scaling symmetry.[5] The importance of this observation lies in the potential usefulness of incorporation of optimization techniques in the practice of the RG methodology. We select three examples: (1) *Gaussian fixed-point criticality in a fluid or in the capillary-wave model of an interface.* We employ the Gaussian distribution for the density fluctuations of the fluid at its critical point, or, equivalently, for the interfacial displacement in the absence of gravity in a capillary-wave model, to evaluate the ordinary Boltzmann–Gibbs–Shannon (BGS) entropy expression. We identify the irrelevant variables and observe their effect in the entropy, and find indications that at the fixed-point where scaling is obeyed at all length scales the entropy is a minimum. (2) *Lévy-like random walls with self-similar cluster formation.*[6–8] We analyze the properties of walks on a lattice described by step distributions with asymptotic power-law decay that may have divergent moments of order two and higher. We corroborate that both the BGS and the non-extensive Tsallis[9] expressions for the entropy decrease when the RG transformation is applied.[5] The existing analogy[10] between a random walk and the Ornstein–Zernike relation for the pair correlation functions in a fluid or magnet is employed to describe the critical phenomena in a lattice gas or Ising model. We find that the anomalous dimension η at criticality is simply related to the index μ of the Lévy distribution, and the parameter q in the Tsallis entropy is used as a measure of the non-extensivity associated to the non-Gaussian fixed point. (3) *Long-ranged bond percolation.*[11] We describe a family of percolation systems on a lattice where each site is connected to all others through a bond occupancy probability distribution that decays asymptotically as a power law of the bond length. An RG transformation of the same form as that appearing in the random walk problem can be seen to apply here too, and therefore we can borrow the entropy expressions derived (and the conclusions drawn from them) in the previous problem. A connection between a geometrical and a thermal system also holds in this case and the properties of the percolation systems would also occur in an equivalent family of Potts models. In our examples we observe that as the RG transformation is applied to the distributions that describe critical systems, as the irrelevant variables decrease in value, the entropy also decreases and is minimal at the fixed point where these variables vanish.

In the following three sections we give details of our analysis of each of these three examples and in the final section we provide a brief summary.

2. GAUSSIAN FIXED-POINT CRITICALITY

Consider the (number) density fluctuation $\delta\rho(\mathbf{r}) \equiv \rho(\mathbf{r}) - \rho_0(\mathbf{r})$ about an equilibrium state $\rho_0(\mathbf{r})$ in a d-dimensional system, and an effective Hamiltonian (divided by $k_B T/2$) of the form

$$H = \int d\mathbf{r}\, d\mathbf{r}'\, C(\mathbf{r}, \mathbf{r}')\, \delta\rho(\mathbf{r})\, \delta\rho(\mathbf{r}'), \tag{1}$$

where $C(\mathbf{r}, \mathbf{r}') \equiv \delta H / \delta\rho(\mathbf{r})\, \delta\rho(\mathbf{r}')$ is simply related to the so-called direct correlation function $c(\mathbf{r}, \mathbf{r}')$ via $C(\mathbf{r}, \mathbf{r}') = \delta_{\mathbf{r}, \mathbf{r}'}/\rho(\mathbf{r}') - c(\mathbf{r}, \mathbf{r}')$. The fluctuation $\delta\rho(\mathbf{r})$ can be decomposed into two terms, $\delta\rho(\mathbf{r}) = -\nabla\rho_0(\mathbf{r}) \cdot \varepsilon(\mathbf{r}) - \rho_0(\mathbf{r})\, \nabla \cdot \varepsilon(\mathbf{r})$, where $\varepsilon(\mathbf{r})$ is the deformation vector. When the fluctuation takes place in an equilibrium state of uniform density ρ_0 the gradient term $\nabla\rho_0(\mathbf{r})$ vanishes, $C(\mathbf{r}, \mathbf{r}') = C(|\mathbf{r} - \mathbf{r}'|)$, and one can write

$$H = \int d\mathbf{r}\, d\mathbf{r}'\, \mathscr{C}(|\mathbf{r} - \mathbf{r}'|)\, \zeta(\mathbf{r})\, \zeta(\mathbf{r}'), \tag{2}$$

where $\mathscr{C}(|\mathbf{r}|) = \rho_0^2 C(|\mathbf{r}|)$ and $\zeta(\mathbf{r}) = \nabla \cdot \varepsilon(\mathbf{r})$, so that $\delta\rho(\mathbf{r}) = -\rho_0 \zeta(\mathbf{r})$. On the other hand, when considering rigid fluctuations of a planar interface $\rho_0(z)$ it is the divergence term $\nabla \cdot \varepsilon(\mathbf{r})$ that vanishes, and $C(\mathbf{r}, \mathbf{r}') = C(|\mathbf{R}|, z, z')$ where $\mathbf{R}$ is a vector parallel to the interface and z and z' are coordinates perpendicular to it. One can write H similarly as

$$H = \int d\mathbf{R}\, d\mathbf{R}'\, \mathscr{C}(|\mathbf{R} - \mathbf{R}'|)\, \zeta(\mathbf{R})\, \zeta(\mathbf{R}'), \tag{3}$$

where now $\mathscr{C}(|\mathbf{R}|) = \int dz\, dz' C(|\mathbf{R}|, z, z')(d\rho_0/dz)(d\rho_0/dz')$ and $\delta(\mathbf{R}, z) = \rho_0(z - \zeta(\mathbf{R})) - \rho_0(z) = -(d\rho_0/dz)\, \zeta(\mathbf{R})$. In Fourier space Eqs. (2) and (3) become (with d replaced by $d-1$ in case of Eq. (3))

$$H = (2\pi)^{-d} \int d\mathbf{k}\, \tilde{\mathscr{C}}(k)\, \tilde{\zeta}(k)\, \tilde{\zeta}(-k), \tag{4}$$

where $k = |\mathbf{k}|$. In the Gaussian model we assume that the moments of $C(|\mathbf{x}|)$, with $\mathbf{x} = \mathbf{r}$ or $\mathbf{x} = \mathbf{R}$, exist and we have $\tilde{\mathscr{C}}(k) = \sum_n C^{(2n)} k^{2n}$ where $C^{(2n)} \equiv (-1)^n (2n!)^{-1} \int d\mathbf{x} |\mathbf{x}|^{2n} C(|\mathbf{x}|)$.

The partition function is given by

$$\Xi = N \prod_{k < \Lambda} \int d\tilde{\zeta}(k)\, d\tilde{\zeta}(-k) \exp(-H[\tilde{\zeta}]), \tag{5}$$

where the normalization constant N is a function of the initial cutoff Λ. The new effective Hamiltonian H' in the RG transformation results from integration of wavenumbers within a shell $\Lambda/b < k < \Lambda$, and because Ξ is Gaussian in $\tilde{\zeta}(k)$ each k term can be integrated out individually to give

$$\Xi = N' \prod_{k < \Lambda/b} \int d\tilde{\zeta}(k)\, d\tilde{\zeta}(-k) \exp(-H'[\zeta]), \tag{6}$$

with $H' = H$ and where N' is the normalization constant for the new cutoff Λ/b. The customary rescaling $k' = bk$ restores the original cutoff and the change in normalization of the order parameter, replacing $\tilde{\zeta}(b^{-1}k')$ by $\tilde{\zeta}'(k') = b^{-(d+2)/2} \tilde{\zeta}(b^{-1}k')$, yields

$$H'[\tilde{\zeta}'] = (2\pi)^{-d} \int d\mathbf{k}'\, \tilde{\mathscr{C}}'(k')\, \tilde{\zeta}'(k')\, \tilde{\zeta}'(-k'), \tag{7}$$

where $\tilde{\mathscr{C}}'(k') = b^2 \tilde{\mathscr{C}}(b^{-1}k')$. Therefore, the moments of $C(|\mathbf{x}|)$ transform as $C'^{(0)} = b^2 C^{(0)}$, $C'^{(2)} = C^{(2)}$, $C'^{(4)} = b^{-2} C^{(4)}$, etc., and we observe that the zeroth moment of $C(|\mathbf{x}|)$ relates to a relevant variable, the second moment remains invariant, and all other higher moments behave as irrelevant variables. At the critical point $C^{(0)}$ vanishes and when the RG transformation is applied repeatedly under this condition the irrelevant variables $b^{-2n+2} C^{(2n)}$, $n \geqslant 2$, tend to zero and vanish at the fixed-point Hamiltonian $H^*[\tilde{\zeta}] = (2\pi)^{-d} \int d\mathbf{k}\, C^{(2)} k^2 \tilde{\zeta}(k)\, \tilde{\zeta}(-k)$.

The probability distribution for the fluctuation $\tilde{\zeta}(k)$ has the Gaussian form

$$P[\tilde{\zeta}(k)] = \Xi^{-1} \exp(-H[\tilde{\zeta}]) = \prod_{\mathbf{k}} (2\pi)^{-1/2}\, \tilde{\mathscr{C}}(k) \exp[-\tilde{\mathscr{C}}(k)\, \tilde{\zeta}(k)\, \tilde{\zeta}(-k)], \tag{8}$$

where the k-component variance is $\tilde{\mathscr{C}}(k)^{-1}$. The entropy associated to P is

$$S = -k_B \sum_{\tilde{\zeta}(k)} P[\tilde{\zeta}(k)] \ln P[\tilde{\zeta}(k)] = -\int d\mathbf{k} \ln \tilde{\mathscr{C}}(k) + \text{constant}, \tag{9}$$

from which we notice that the entropy difference $\Delta S = S' - S$ between successive applications of the RG transformation would be negative if $\tilde{\mathscr{C}}'(k) > \tilde{\mathscr{C}}(k)$, and positive otherwise. Thus, provided the contribution to $\tilde{\mathscr{C}}(k)$ from the irrelevant variables $C^{(2n)}$, $n \geqslant 2$, is always negative, i.e., $\sum_{n=2} C^{(2n)} k^{2n} < 0$, the fixed-point entropy would be a minimum with respect to the entropy of all other critical points $C^{(0)} = 0$ that define the so-called critical hypersurface. Naturally, if $\sum_{n=2} C^{(2n)} k^{2n} > 0$ the fixed-point entropy would be maximum.

In the case of the critical point of a uniform fluid phase $C^{(0)} = \rho_0 (k_B T K_T)^{-1}$ where K_T is the isothermal compressibility, and for an intermolecular interaction potential consisting of a hard core repulsion when $|\mathbf{r}| \leqslant \sigma$ followed by an attractive term $u(|\mathbf{r}|) < 0$ when $|\mathbf{r}| > \sigma$ one has, in mean-field approximation, that $C^{(2n)} = (-1)^n (2n!)^{-1} \rho_0^2 (k_B T)^{-1} \int d\mathbf{r} |\mathbf{r}|^{2n} u(|\mathbf{r}|)$, $n \geqslant 1$. The sign of $\sum_{n=2} C^{(2n)} k^{2n}$ is determined by the competition between negative and positive moments and the decay of $u(|\mathbf{r}|)$ for large $|\mathbf{r}|$ is decisive. The negative moments have lower orders that those for the positive and this suggests they are dominant. For instance, if we consider the one-dimensional Kac potential $u(|r|) = -a\gamma \exp(-\gamma |r|)$, $a > 0$, in the limit $\gamma \to 0$, we obtain $C^{(0)} = \rho_0 (1 - \sigma \rho_0)^{-2} - 2a\rho_0^2 (k_B T)^{-1}$, $C^{(2)} = 2a\rho_0^2 (k_B T)^{-1}$, and $\sum_{n=2} C^{(2n)} k^{2n} = -2a\rho_0^2 (k_B T)^{-1} k^4 (1 + k^2)^{-1} < 0$.

For a liquid-vapor planar interface under a gravitational field $C^{(0)} = (k_B T)^{-1} \Delta \rho m g$ (where $\Delta \rho$ is the density difference between the two phases, m is the molecular mass and g the gravitational acceleration), $C^{(2)} = (k_B T)^{-1} \gamma$ and $C^{(4)} = (k_B T)^{-1} \kappa$, where γ is the surface tension and κ the bending rigidity.[12] The function $\mathscr{C}(k)$ has been determined recently[13] from accurate Molecular Dynamics simulations of the planar interface for Lennard-Jones spherical particles, and κ was found to be unequivocally negative, a result that supports the property that the fixed-point entropy is a minimum also in this case. Here the fixed-point Hamiltonian corresponds to the ordinary capillary-wave model that considers only surface tension as the restoring force for thermal distortions in the absence of gravity, whereas the extended capillary-wave model[12] corresponds to other critical-point Hamiltonians where the presence of additional restoring forces, defined by the higher moments $C^{(2n)}$, $n \geqslant 2$, appear as irrelevant variables.

3. RANDOM WALKS WITH SELF-SIMILAR CLUSTER FORMATION

We recall first some features of a symmetric one-dimensional random walk on a lattice[6–8] that exhibits a scaling property such that under appropriate conditions the trajectories consist of a hierarchy of self-similar clusters of visited sites. The walks are generated by a distribution for single steps $p^{\mu}_{\infty}(l)$ of the form

$$p^{\mu}_{\infty}(l)=\frac{A_{\infty}}{2}\sum_{n=0}^{\infty}a^{-n}(\delta_{l,\,-b^n}+\delta_{l,\,b^n}), \tag{10}$$

where $a>1$ and $b>1$. The allowed steps have unevenly spaced step lengths b^n and occur with probabilities proportional to a^{-n}. Equation (10) can be rewritten as the power-law $p^{\mu}_{\infty}(l)=A_{\infty}|l|^{-\mu}$, $A_{\infty}=1-a^{-1}$, $\mu\equiv\ln a/\ln b$, and it can be seen that when $\mu<2$ the mean-squared displacement per jump diverges. The structure function $\lambda^{\mu}_{\infty}(k)=\sum_l p^{\mu}_{\infty}(l)\exp(ikl)$ is the continuous non-differentiable function of Weierstrass

$$\lambda^{\mu}_{\infty}(k)=A_{\infty}\sum_{n=0}^{\infty}a^{-n}\cos(b^n k), \tag{11}$$

and its non-analytic small-k behavior was demonstrated[6, 7] to arise from an infinite sum of regular terms obtained by iteration of the scaling equation

$$\lambda^{\mu}_{\infty}(k)=a^{-1}\lambda^{\mu}_{\infty}(k)+A_{\infty}\cos k. \tag{12}$$

When $\mu\leqslant 2$ the singular part of $\lambda^{\mu}_{\infty}(k)$ is of the form $Q(k)|k|^{\mu}$ with $Q(k)$ periodic in $\ln|k|$ with period $\ln b$. The random walk was interpreted[6, 7] as having an effective dimension $d_{eff}=3-\mu$ that exceeds the available spatial dimension $d=1$ if $\mu<2$. The difference in dimensions $d_{eff}-d=2-\mu$ determines whether the walk is persistent $d_{eff}-d\leqslant 1$ (the walker is certain to return to any site because the set of sites visited by the walk cover the lattice) or transient $d_{eff}-d>1$ (the return probability of the walker is less than unity because the set of sites in the walk does not fill the entire lattice). Thus, the walk is transient and trajectories display self-similar clusters when $\mu<1$. The exponent μ was identified with the fractal dimension of the set of sites visited by the walks and the connection with the Lévy distributions was exhibited.[6, 7] The walk in Eq. (10) can be generalized straightforwardly to square and simple cubic lattices.

For our purposes we enlarge the random walk problem in Eq. (10) and consider a family of walks such that a class of them would be attracted under the RG transformation to the original walk as a fixed point.[5] Therefore, we generalize the expression for the distribution of single steps $p_r(l)$ to be

$$p_r(l)=\frac{A_r}{2}\sum_{n=0}^{r}a_n(\delta_{l,\,-b^n}+\delta_{l,\,b^n}), \tag{13}$$

the step lengths b^n, $b > 1$, have been maintained but now the probabilities assigned to them are proportional to arbitrary positive numbers a_n. Also a range for the step lengths b^r has been introduced, but the possibility $r \to \infty$ is included; A_r normalizes $p_r(l)$, i.e., $A_r^{-1} = \sum_{n=0}^{r} a_n$. The elementary RG transformation $a'_n \equiv R[a_n] = a a_{n+1}$ can be applied to our family of walks. This transformation maps the sites $l = b^{n+1}$ into the sites $l' = b^n$ (eliminating intermediate lattice space between allowed step lengths) and renormalizes the step probability by a restoring factor a. It is clear that the Weierstrass walk $p_{\infty}^{\mu}(l)$ and the simple nearest-neighbor step walk $p_0(l)$ are both fixed points of R. The first one is non-trivial in the sense that it is associated to an infinite-ranged step distribution that can be reached via de RG transformation only from other infinite-ranged step distributions $p_{\infty}(l)$ required to approach asymptotically the condition $a_n = a^{-n}$, $n \to \infty$. The distributions $p_{\infty}(l)$ make up the "critical hypersurface"' and the quantities $\alpha_n \equiv a_n - a^{-n}$ are the irrelevant variables that vanish as R is repeatedly applied. The other fixed point $p_0(l)$ is trivial since it is generated by the application of the RG transformation to any "noncritical" finite-ranged $p_r(l)$, $r < \infty$.

The entropy of the step distribution $p_r(l)$ along two representative types of RG trajectories has been evaluated.[5] These are: a noncritical trajectory starting with a truncated power-law distribution

$$p_r^{(1)}(l) = (A_r/2) \sum_{n=0}^{r} a^{-n}(\delta_{l,\,-b^n} + \delta_{l,\,b^n}) \tag{14}$$

that flows under R into the trivial fixed point $p_0(l)$, and a critical trajectory with a starting infinite-ranged distribution

$$p_m^{(2)}(l) = (A_m/2) \sum_{n=0}^{m} a_n(\delta_{l,\,-b^n} + \delta_{l,\,b^n}) + (A_m/2) \sum_{n=m+1}^{\infty} a^{-n}(\delta_{l,\,-b^n} + \delta_{l,\,b^n}) \tag{15}$$

that flows under R into the non-trivial fixed point $p_{\infty}^{\mu}(l)$. We quote below the results obtained for the entropy of the walks in Eqs. (14) and (15) where the summations over the lattice sites were done over $|l|$ instead of l. In this way the expressions obtained are independent of lattice coordination number and also, as we shall see below, they can be used immediately for the bond percolation problem described in the next section. For $p_r^{(1)}(l)$ the BGS expression $S_1 \equiv -k_B \sum_{|l|} p(l) \ln p(l)$ yields

$$k_B^{-1} S_1^r[p^{(1)}] = \ln \frac{1-\epsilon^{r+1}}{1-\epsilon} - \frac{\epsilon \ln \epsilon}{1-\epsilon} + \frac{(r+1)\,\epsilon^{r+1} \ln \epsilon}{1-\epsilon^{r+1}}, \tag{16}$$

for all μ with $\epsilon = a^{-1}$. Whereas the generalized Tsallis entropy[9] $S_q \equiv k_B(q-1)^{-1}\{1-\sum_{|l|}[p(l)]^q\}$, that is non extensive for $q \neq 1$ but reduces to the customary extensive expression when $q=1$,[9] gives

$$S_q^r[p^{(1)}] = \frac{k_B}{q-1}\left[1 - \frac{(1-\epsilon)^q}{1-\epsilon^q}\frac{1-\epsilon^{q(r+1)}}{(1-\epsilon^{r+1})^q}\right], \tag{17}$$

again for all μ. The fixed point $p_0(l)$ has a vanishing entropy $S_q^0 = 0$ for all q, and by taking the limit $S_q^\infty = \lim_{r\to\infty} S_q^r$ we obtain for the non-trivial fixed point

$$S_q^\infty = \frac{k_B}{q-1}\left[1 - \frac{(1-\epsilon)^q}{1-\epsilon^q}\right] \tag{18}$$

with

$$S_1^\infty = k_B\left[\ln\frac{1}{1-\epsilon} - \frac{\epsilon\ln\epsilon}{1-\epsilon}\right]. \tag{19}$$

For all $q \geqslant 1$ and all $r' > r > 0$ we find (since $\epsilon^{q(r+1)} < q\epsilon^{r+1}$, $0<\epsilon<1$) that $S_q^0 < S_q^r < S_q^{r'} < S_q^\infty$, that is, the entropy is a monotonously increasing function of the step length range r of the distribution $p_r^{(1)}(l)$, being a maximum for S_q^∞ and a minimum for S_q^0. Because each time the RG transformation $a_n' \equiv R[a_n] = aa_{n+1}$ is applied to the walk distribution $p_r^{(1)}(l)$ it has precisely the effect of shifting the value of r to $r-1$ in Eq. (14), we notice that the entropy along the RG flow is monotonously decreasing and vanishes at the trivial fixed point.

For $p_m^{(2)}(l)$ we present results when the deviation from $p_\infty^\mu(l)$, $\delta p_m(l) \equiv (A_m/2)\sum_{n=0}^m \alpha_n(\delta_{l,-b^n} + \delta_{l,b^n})$, is small, i.e., when only terms linear in $\delta p_m(l)$ are retained. The expression $S_1 \equiv -k_B\sum_{|l|} p(l)\ln p(l)$ yields

$$k_B^{-1} S_1^m[p^{(2)}] = \ln\frac{1-\delta_m}{1-\epsilon} - \left[\frac{\epsilon(1-\delta_m)}{1-\epsilon} - \gamma_m\right]\ln\epsilon - \delta_m, \tag{20}$$

for all μ, where $\delta_m \equiv \sum_l \delta p_m(l) = A_m\sum_{n=0}^m \alpha_n$ and $\gamma_m \equiv A_m\sum_{n=0}^m n\alpha_n$. The Tsallis expression $S_q \equiv k_B(q-1)^{-1}\{1-\sum_{|l|}[p(l)]^q\}$ under the same condition gives

$$S_q^m[p^{(2)}] = \frac{k_B}{q-1}\left[1 - \frac{(1-\epsilon)^q}{1-\epsilon^q}(1-\delta_m)^q - q(1-\epsilon)^{q-1}(1-\delta_m)^{q-1}\gamma_m\right], \tag{21}$$

again for all μ, where now $\gamma_m \equiv A_m \sum_{n=0}^{m} \epsilon^{(q-1)n} \alpha_n$. If we limit the departure of the magnitude for the step probability coefficients a_n, $n \leqslant m$, from the fixed-point power law value a^{-n} (i.e., if we bound the departure of the irrelevant variables $\alpha_n = a_n - a^{-n}$ from zero) we can prove that the entropy decreases monotonously as $\delta_m \to 0$ and is a minimum at the fixed point $\delta_m = 0$. With this purpose we adopt the constraint

$$\sum_{n=0}^{m} n\alpha_n = C_1 \sum_{n=0}^{m} \alpha_n, \tag{22}$$

when $q = 1$, and chose C_1 to be $C_1 = \epsilon(1-\epsilon)^{-1}$ since then the constraint contains as a special case the fixed point condition $a_n = aa_{n+1}$. A generalization of this procedure for $q > 1$ leads to a constraint of the form

$$\sum_{n=0}^{m} \epsilon^{(q-1)n} \alpha_n = C_q \sum_{n=0}^{m} \alpha_n, \tag{23}$$

where the constant C_q above is chosen to be $C_q = (1-\epsilon)(1-\epsilon^q)^{-1}$ to include $a_n = aa_{n+1}$ also as a special case. We find for all $q \geqslant 1$ and all $m > 0$ that $S_q^m > S_q^\infty$, therefore, the non-trivial fixed point distribution $p_\infty^\mu(l)$ has an entropy S_q^∞ smaller than that for any other infinite-ranged step distribution $p_m^{(2)}(l)$. It can be readily verified that S_q^m, for all $q \geqslant 1$, decreases monotonously as $|\delta_m| \to 0$. Every time the RG transformation $a'_n \equiv R[a_n] = aa_{n+1}$ is applied to the walk $p_m^{(2)}(l)$ it shifts the value of m to $m-1$ in Eq. (15), and therefore we obtain that when all the $\alpha_n \geqslant 0$, or all the $\alpha_n \leqslant 0$, the entropy along the (critical) RG flow is monotonously decreasing and attains a minimum at the nontrivial fixed point. When the α_n are of differing signs the entropy along the RG flow decreases monotonously or not towards the fixed-point minimum according to whether $|\delta_m|$ decreases monotonously or not towards zero as $m \to 0$.

The mean-square displacement $\langle l^2 \rangle_1^\infty = \sum_l l^2 p_\infty^\mu(l)$ diverges when $\mu \leqslant 2$, but $\langle l^2 \rangle_q^\infty = \sum_l l^2 [p_\infty^\mu(l)]^q$ is finite when $\mu \leqslant 2$ with $q > 1$. The limiting value of q for the convergence of $\langle l^2 \rangle_q^\infty$ is $q = 1$ for $\mu > 2$, and $q = 2/\mu$ for $\mu \leqslant 2$ and this choice of the parameter q provides a convenient measure of non-extensivity at the critical point. Thus, the Gaussian non-fractal behavior obtained when $\mu > 2$ is extensive, whereas the Lévy-type fractal behavior for $\mu \leqslant 2$ is increasingly non-extensive as the dimension difference $d_{eff} - d = 2 - \mu$ departs from zero.

The results obtained for the random walk can be applied to the critical point of a simple fluid or magnet defined on the same lattice. The Ornstein–Zernike equation $h(l) = c(l) + \rho \sum_{l'} c(l')\, h(l-l')$ relating the total pair correlation $h(l)$ with the direct correlation function $c(l)$ of the fluid of density ρ can be put into correspondence[10] with the equation for the random walk generating function $P(l; z) - z \sum_{l'} p(l') P(l-l'; z) = \delta_{l,0}$ where $P(l; z) = \sum_n P_n(l) z^n$, and where $P_n(l)$ is the probability of occupancy of site l after n steps.[14] The correlations for $l \neq 0$ are given by $c(l) = wzp(l)$ and $\rho^2 h(l) = w^{-1} P(l; z)$, where $w = \rho^{-1}(1-\rho)^{-1} P(0; z)$. The power-law decay given to $c(l)$ by an infinite-ranged $p_\infty(l)$ introduces criticality in the system and the divergence of the susceptibility $\chi = (1-\rho)[\rho P(0; z)(1-z)]^{-1}$[10] indicates that the critical point is attained when $z = 1$. One can easily verify through the equivalence between $h(l)$ and $P(l; z)$ that the anomalous dimension exponent η is $\eta = d_{eff} - d = 2 - \mu$. In Fourier space $\tilde{h}(k) \sim [1 - z\lambda_\infty(k)]^{-1}$ and at the critical fixed point $\tilde{h}(k) \sim |k|^{-\mu}$ when $z = 1$ and $k \to 0$.

4. LONG-RANGED BOND PERCOLATION

The entropic properties for the random walk presented above can be seen to apply also when the RG approach is employed in the study of a closely analogous percolation problem. This is a bond percolation problem in a one-dimensional lattice in which each site is connected to all other sites through bonds that occur with a prescribed probability p_{ij} where i and j are the positions of two sites separated by a distance $l = |j - i|$. A relevant particular case, studied recently,[11] is that in which the bond occupancy probabilities are given the power-law form $p_{ij} = pl^{-\alpha}$, where $0 \leqslant p \leqslant 1$ is the occupancy probability for bonds between first neighbors and $\alpha \geqslant 0$. It has been found via numerical simulations[11] that this specific example displays three regimes with differing percolation threshold p_c. In the first, $p_c = 1$ when $2 \leqslant \alpha < \infty$ (the $\alpha \to \infty$ limit corresponds to the ordinary, first-neighbor, problem), in the second, $0 < p_c < 1$ when $1 < \alpha < 2$, and in the third, $p_c = 0$ when $0 \leqslant \alpha \leqslant 1$. An interesting cross-over from extensive to non-extensive behavior takes place for $\alpha = 1$ and the introduction of a re-scaled threshold probability p_c^* no longer vanishes when $0 \leqslant \alpha \leqslant 1$ and presents continuous behavior across the entire interval $0 \leqslant \alpha \leqslant 2$.[11] rescaling of p appears necessary because for sufficiently slow decay of the power law, i.e., when $\alpha \leqslant 1$, the number of bonds in the system $\sum_{i,j} p_{ij}$ are no longer proportional to the number of lattice sites.

To facilitate immediate use of all the results in the previous section we restrict the allowed bond lengths to be of sizes b^n, $b > 1$, with probabilities proportional to a_n, and introduce a range b^r for these bond lengths. This way the (normalized) bond occupancy distribution is given by

$$p_r(l) = A_r \sum_{n=0}^{r} a_n \delta_{l, b^n}, \tag{24}$$

where $A_r^{-1} = \sum_{n=0}^{r} a_n$ and $a_0 = p$, and the same RG transformation $a'_n \equiv R[a_n] = a a_{n+1}$ can be employed here. Thus, the power-law $p_{ij} = p l^{-\alpha}$ studied in ref. 11 can be recognized as being the nontrivial fixed-point distribution for our family of percolation systems, and the exponent α as the Lévy-walk exponent $\mu \equiv \ln a/\ln b$ (that is $a_n = a^{-n}$, $a > 1$, $r \to \infty$), also the first-neighbor problem is identified as the trivial fixed point. Our main result holds, i.e., along the RG flow the entropy is monotonously decreasing and attains a minimum at the fixed point distributions. We can identify the regime cross-over at $\alpha = 2$ as the transition from extensive Gaussian behavior to non-extensive Lévy-like behavior. When $\alpha \leqslant 2$ the second (and higher) order moments of p_{ij} become divergent and as before we can introduce the dimension difference $d_{eff} - d = 2 - \alpha$ and the entropy index $q = 2/\alpha$ as measures of the degree of non-extensivity. The second regime cross-over at $\alpha = 1$ can be identified as the onset of percolation clusters that acquire the self-similar fractal property, for $\alpha < 1$ the set of sites that form the percolation clusters (of fractal dimension α) do not cover the entire lattice. Finally, we note that the distribution of bond lengths at the nontrivial fixed point has the same non-analytic features responsible for the non-Gaussian behavior of the walk defined by Eq. (10), features that have been shown[6, 7] to be analogous to the singular behavior displayed by thermodynamic properties in ordinary critical phenomena. Specifically, the Fourier transform of p_{ij} satisfies Eq. (12), a scaling property equivalent to that of the transformation equation for the free energy of a spin system under the renormalization group.[6, 7] A well-known equivalence exists between a bond percolation problem and a single-state Potts model defined on the same lattice[15] where the bond occupation probabilities p_{ij} and the Potts coupling constants J_{ij} relate as $p_{ij} = 1 - \exp(J_{ij}/k_B T)$.

5. SUMMARY

In summary, we have considered three examples where self-similarity under rescaling takes place, Gaussian criticality, random walks with long-tailed step distributions, and long-ranged bond percolation. These are amongst the simplest systems for which the RG approach can be applied and offer particularly transparent RG transformations, flow properties and fixed points, and we have taken advantage of this explicitness to probe on the possible variational properties of this method. In all cases we found evidence for a decreasing entropy function along the RG flows that becomes a minimum at the fixed points. The entropies are evaluated for pertinent distributions as they are transformed by the RG. These are: the distribution for order-parameter fluctuations in the Gaussian model, the distribution of single steps in the random walk, and the distribution of bond occupancies in the percolation problem. In the Gaussian model we considered both bulk criticality and interfacial capillary-waves and we described the role of the moments (higher than the second) of the direct correlation function as RG irrelevant variables. By construction there is a straightforward equivalence between the random walk and the percolation problems described and they are seen to share the same properties, one of which is fractal self-similar clustering. When the decay of the fixed-point power law distribution becomes sufficiently slow departure from Gaussian behavior occurs and non-extensive behavior of Lévy-type long-tailed distributions is obtained. The sharp onset of this non-analytic regime is equivalent to that taking place at the borderline dimensionalities below which classical theories breakdown. Indeed, we have seen that in the statistical-mechanical analogs of these problems this behavior gives rise to a non-vanishing dimensional anomaly. Interestingly, the extensions described here of the previously studied[6, 7] Levy-like lattice random walk have revealed that this walk, with pure power-law step distribution and with structure function given by the Weierstrass function, corresponds to a RG non-trivial fixed point. In all three examples we observe that along the critical RG transformation flows, over which the irrelevant variables decrease in value, the entropy also decreases and becomes a minimum at the fixed point where these variables vanish. The links we have exhibited amongst the various properties of scaling symmetry suggest that the variational technique of optimal entropy may be of practical importance to the RG applications.

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